Exercise 9.6.2

Solve the wave equation, Eq. (9.89), subject to the indicated conditions.

Determine $\psi(x,t)$ given that at $t = 0$ $\psi_0(x) = \delta(x)$ (Dirac delta function) and the initial time derivative of ψ is zero.

Solution

The initial value problem to solve is as follows.

$$\begin{aligned} &\psi_{tt} = c^2\psi_{xx}, \quad -\infty < x < \infty, \ -\infty < t < \infty \\ &\psi(x,0) = \delta(x) \\ &\psi_t(x,0) = 0 \end{aligned}$$

Since the wave equation is over the whole line ($-\infty < x < \infty$), it can be solved by operator factorization. Bring $c^2\psi_{xx}$ to the left side.

$$\frac{\partial^2\psi}{\partial t^2} - c^2\frac{\partial^2\psi}{\partial x^2} = 0$$

Factor the operator.

$$\left(\frac{\partial^2}{\partial t^2} - c^2\frac{\partial^2}{\partial x^2}\right)\psi = 0$$

$$\left(\frac{\partial}{\partial t} + c\frac{\partial}{\partial x}\right)\left(\frac{\partial}{\partial t} - c\frac{\partial}{\partial x}\right)\psi = 0$$

$$\left(\frac{\partial}{\partial t} + c\frac{\partial}{\partial x}\right)\left(\frac{\partial\psi}{\partial t} - c\frac{\partial\psi}{\partial x}\right) = 0$$

Let u be the quantity in the second set of parentheses.

$$\left(\frac{\partial}{\partial t} + c\frac{\partial}{\partial x}\right)u = 0$$

$$\frac{\partial u}{\partial t} + c\frac{\partial u}{\partial x} = 0$$

As a result of factoring the operator, the wave equation has reduced to a system of first-order PDEs.

$$\left.\begin{aligned} \frac{\partial\psi}{\partial t} - c\frac{\partial\psi}{\partial x} = u \\ \frac{\partial u}{\partial t} + c\frac{\partial u}{\partial x} = 0 \end{aligned}\right\}$$

The differential of a function of two variables $h = h(x,t)$ is defined as

$$dh = \frac{\partial h}{\partial t}\,dt + \frac{\partial h}{\partial x}\,dx.$$

Divide both sides by dt to obtain the fundamental relationship between the total derivative of h and the partial derivatives of h.

$$\frac{dh}{dt} = \frac{\partial h}{\partial t} + \frac{dx}{dt}\frac{\partial h}{\partial x}$$

In light of this, the PDE for u reduces to the ODE,

$$\frac{du}{dt} = 0, \tag{1}$$

along the characteristic curves in the xt-plane that satisfy

$$\frac{dx}{dt} = c, \quad x(\xi, 0) = \xi, \tag{2}$$

where ξ is a characteristic coordinate. Integrate both sides of equation (2) with respect to t to solve for $x(\xi, t)$.

$$x = ct + \xi$$

Now integrate both sides of equation (1) with respect to t.

$$u(x, \xi) = f(\xi)$$

f is an arbitrary function of the characteristic coordinate ξ. Eliminate ξ in favor of x and t.

$$u(x, t) = f(x - ct)$$

Consequently, the PDE for ψ becomes

$$\frac{\partial \psi}{\partial t} - c\frac{\partial \psi}{\partial x} = f(x - ct).$$

It reduces to

$$\frac{d\psi}{dt} = f(x - ct) \tag{3}$$

along the characteristic curves in the xt-plane that satisfy

$$\frac{dx}{dt} = -c, \quad x(\eta, 0) = \eta, \tag{4}$$

where η is another characteristic coordinate. Integrate both sides of equation (4) with respect to t to solve for $x(\eta, t)$.

$$x = -ct + \eta$$

Now integrate both sides of equation (3) with respect to t.

$$\psi(x, \eta) = \int^t f(x - cs)\, ds + G(\eta)$$

G is an arbitrary function of the characteristic coordinate η. Make the substitution $r = x - cs$ in the integral.

$$\begin{aligned}\psi(x, \eta) &= \int^{x-ct} f(r)\left(-\frac{dr}{c}\right) + G(\eta) \\ &= F(x - ct) + G(\eta)\end{aligned}$$

F is the integral of $-f/c$, another arbitrary function. Therefore, since $\eta = x + ct$,

$$\psi(x, t) = F(x - ct) + G(x + ct).$$

This is the general solution of the wave equation. Now apply the initial conditions to determine F and G.

$$\begin{aligned}\psi(x,0) &= F(x) + G(x) = \delta(x) \\ \psi_t(x,0) &= -cF'(x) + cG'(x) = 0\end{aligned}$$

Differentiate both sides of the first equation with respect to x and multiply both sides of it by c.

$$\begin{aligned}cF'(x) + cG'(x) &= c\delta'(x) \\ -cF'(x) + cG'(x) &= 0\end{aligned}$$

Add both sides of each equation to eliminate F'.

$$2cG'(x) = c\delta'(x)$$

Divide both sides by $2c$.

$$G'(x) = \frac{1}{2}\delta'(x)$$

Integrate both sides with respect to x, setting the constant of integration to zero.

$$G(x) = \frac{1}{2}\delta(x)$$

So then

$$F(x) + G(x) = \delta(x) \quad \rightarrow \quad F(x) + \frac{1}{2}\delta(x) = \delta(x) \quad \rightarrow \quad F(x) = \frac{1}{2}\delta(x).$$

What we have actually solved for are $F(w)$ and $G(w)$, where w is any expression we choose.

$$\begin{aligned}F(x - ct) &= \frac{1}{2}\delta(x - ct) \\ G(x + ct) &= \frac{1}{2}\delta(x + ct)\end{aligned}$$

As a result,

$$\begin{aligned}\psi(x,t) &= F(x - ct) + G(x + ct) \\ &= \frac{1}{2}\delta(x - ct) + \frac{1}{2}\delta(x + ct).\end{aligned}$$

Therefore,

$$\psi(x,t) = \frac{1}{2}\left[\delta(x - ct) + \delta(x + ct)\right].$$